The codex32 *Mathematical Companion*

Pearlwort Snead

Contents

1 Mathematical Preliminaries

In this first section we give a crash course in field theory. Readers familiar with this material should have no problem skipping or skimming this section, at least up to Section 1.5. Readers who are completely unfamiliar with this material are unlikely to be able to follow the condensed exposition, and are encouraged to consult standard algebra textbooks (e.g. the one by Dummit and Foote) or Wikipedia.

1.1 Fields and $\mathbb{F}_2$

Consider the integers modulo 2. This is a set consisting of two equivalence classes, the evens and the odds, which hereafter we will refer to as 0 and 1. This set is a **field**, which means that when we define addition and multiplication in the obvious way, it satisfies the following axioms:

1. The set is closed under addition; addition is associative, commutative, has an identity element 0, and all elements have additive inverses. In other words it is an **abelian group** under addition.
2. Similarly, the nonzero elements form an abelian group under multiplication, with identity 1.
3. The **distributive law** holds, which means that $a(b+c)$ always equals $ab+ac$.

We refer to this field as $\mathbb{F}_2$. For any field, we refer to its nonzero elements as the **multiplicative group** of the field. We observe that the multiplicative group of $\mathbb{F}_2$ has only the identity element.

1.2 Polynomial Rings

Since $\mathbb{F}_2$ has only two elements, it is hard to do interesting algebra on it. But it is a fact that, by adjoining a formal symbol x to a field, we can obtain a much bigger (in fact, countably infinite) set of *polynomials* in $\mathbb{F}_2$. We denote this set $\mathbb{F}_2[x]$ and call it the **polynomial ring** of the field.

Formally, the set of polynomials is defined as

$$\left\{ \sum_{i=0}^{n} a_i x^i : n \in \mathbb{N} \cup \{0\}, a_i \in \mathbb{F}_2 \right\}$$

A ring, for our purposes, is defined the same way as a field except that we do not require multiplication to be invertible. In particular, we will only consider commutative rings. It is easy to check that the polynomial ring, endowed with addition and multiplication in the obvious ways, is in fact a ring.

For a polynomial of the form $\sum_{i=0}^{n} a_i x^i$ with $a_n \neq 0$ we refer to n as the **degree** of the polynomial. It is an elementary fact that the product of polynomials has degree equal to the sum of the degrees of the factors.

We refer to polynomials of degree 0 as **constant polynomials**. It is also a fact that a polynomial has a multiplicative inverse, i.e. it is a **unit**, if and only if it is nonzero constant.

A polynomial r is **irreducible** if, whenever it is written as the product of two polynomials $r = pq$, either p or q is a unit (i.e. degree 0). Otherwise, r is **reducible**.

1.3 Quotient Fields

Just like we can consider the integers modulo some integer n, thus obtaining n equivalence classes which inherit (roughly) the original ring structure of the integers, we can consider a polynomial ring modulo some polynomial p. In this case, we will get m^n equivalence classes, where m is the number of elements in the underlying field and n is the degree of the polynomial.

We call the set of equivalence classes a **quotient ring**, and its addition and multiplication are defined in the obvious way.

Just like in the integer case, if our polynomial p can be factored into nonconstant polynomials as $p = p_1 p_2$, then the images of p_1 and p_2 in the quotient ring will be nonzero but satisfy $p_1 p_2 = 0$. In other words they are **zero divisors** and imply that multiplication in the ring is not invertible.

We do not like zero divisors, so from here on out we will be sure to mod out our polynomial ring only by irreducible polynomials. It is a fact that the resulting quotient ring will then be a field, and we term it a **quotient field**. It is a fact that $x^5 + x^3 + 1$ is irreducible in $\mathbb{F}_2$, so that $\mathbb{F}_2/(x^5 + x^3 + 1)$ is a quotient field with 32 elements. The original field, $\mathbb{F}_2$, we refer to as the **base field**.

In this field the object x is a field element with a distinct identity and algebraic properties, so we rename it α to preserve the symbol x to be an indeterminate used for writing polynomials.

For any element δ in the quotient field, we can talk about its **minimal polynomial** over the base field. This is a monic polynomial (one whose highest-degree coefficient is 1) over the base field, of minimal degree, such that δ is a root when the polynomial is considered over the extension field. It is a fact of field theory that for any element δ, a unique such minimal polynomial exists. We sometimes refer to the **degree** of δ as being the degree of its minimal polynomial.

Whenever we walk about minimal polynomials or degrees of field elementns, it is understood that we are considering the elements relative to some base field, but it will always be clear from context what this base field is, so that we can use these terms unambiguously.

It is a fact that, for this specific polynomial, that α is a **generator** of the multiplicative group of the quotient field, meaning that the field in its entirety is equal to

$$\left\{\alpha^i : i \in \{0, 1, \ldots, 30\}\right\} \cup \{0\}.$$

We observe that the order of the multiplicative group is 31, a prime, and therefore every element of the group except 1 is a generator of the group. Furthermore there are no nontrivial proper subgroups. These are elementary facts of group theory.

We refer to this new field as $\mathbb{F}_{32}$. It is a fact of field theory that all groups with 32 elements are isomorphic to this one, which justifies the name. But bear in mind that, for our purposes, the field was constructed as $\mathbb{F}_2[x]/(x^5 + x^3 + 1)$ and has a distinguished generator α which is a root of that polynomial.

1.4 Vector Spaces

We observe that $\mathbb{F}_{32}$ is a **vector space** over $\mathbb{F}_2$. A vector space V over a field $\mathbb{F}$ is defined by the following axioms:

1. V is an abelian group with operation $+$ and identity 0_V.
2. $(a+b)v = av + bv$ and $a(u+v) = au + av$ for all $a, b \in \mathbb{F}$ and $u, v \in V$.

We refer to a finite sum of the form $\sum_i f_i v_i$ with $f_i \in \mathbb{F}$ and $v_i \in V$ as a **linear combination**. We observe that every element of $\mathbb{F}_{32}$ is a linear combination of the elements $\{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}$, and that no smaller set of elements has this property. We call such a set a **basis** for $\mathbb{F}_{32}$.

1.5 Lagrange Interpolation and Shamir's Secret Sharing

Let $\mathbb{F}$ be a field and p a polynomial of degree n in $\mathbb{F}[x]$. It is a standard theorem of algebra that p's value on all points of $\mathbb{F}$ is implied by its values on any $n+1$ distinct points.

As discovered by Edward Waring in 1779[1], and later by Joseph-Louis Lagrange in 1795[2][3], it is actually possible to compute the value of a degree n polynomial at a field element x explicitly in terms of its values at $n+1$ given distinct points x_i.

Specifically, suppose that $p(x_i) = y_i$. Then

$$p(x) = \sum_{i=1}^{n+1} y_i \ell_i(x) \tag{1}$$

where ℓ_i is determined entirely by the x_i's, as

$$\ell_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

There are several very interesting observations to be made here:

1. First, for a fixed set of x_i's, we see that the vector space of n-degree polynomials over $\mathbb{F}$ is spanned by the set $\{\ell_i(x)\}$. Since there are $n+1$ polynomials and this space has dimension $n+1$ (an obvious basis for it is $\{1, x, x^2, \ldots, x^n\}$), this means that the set $\{\ell_i(x)\}_i$ forms a basis for this space.
2. Further, these basis polynomials satisfy the equality $\sum_i \ell_i(x) = 1$. (One way to see this is by using equation (1) to interpolate the constant one polynomial.)

   This means that equation (1) is an **affine combination** of the y_i's, a strengthening of the familiar notion of linear combination. This property will become important, as we will see.

[1]Waring, Edward (1779). "Problems concerning interpolations". *Philosophical Transactions of the Royal Society.* 69: 59–67. doi:10.1098/rstl.1779.0008.

[2]Lagrange, Joseph-Louis (1795). "Leçon Cinquième. Sur l'usage des courbes dans la solution des problèmes". *Leçons Elémentaires sur les Mathématiques*

[3]Both citations taken from Wikipedia's "Lagrange Interpolation" page, March 2023.

3. If we further fix x, we see that knowing p's evaluation at every x_i is sufficient to determine $p(x)$, while knowing any fewer evaluations provides zero information about x: suppose for example that y_n is unknown. Then by a suitable choice of y_n in (1) we can cause $p(x)$ to take any of the $|\mathbb{F}|$ possible values.

Putting these facts together, we obtain **Shamir's Secret Sharing Scheme** (SSSS) for splitting a secret element of $\mathbb{F}$ into up to $|\mathbb{F}| - 1$ shares, such that a fixed threshold number k of them are sufficient to reconstruct the secret:

1. First, fix an index $s \in \mathbb{F}$ to be the secret index.

2. Generate a random $(k-1)$-degree polynomial p by choosing k random values and assigning them to be the evaluation of p at specific points $x_i \in \mathbb{F}$.

   (If the secret is known beforehand, then fix $p(s)$ to be the secret and generate $k-1$ other evaluations of p randomly.)

3. Distribute the points x_i along with their evaluations $p(x_i)$ to multiple parties.

4. Then if any k of them come together, they can use equation (1) to reconstruct the secret $p(s)$.

We call the k randomly generated values **initial shares** and every other evaluation of p a **derived share**.

There are several interesting observations here:

- If we have a sequence $F_x = \{f_i\}$ of elements of $\mathbb{F}$, we can use SSSS in parallel on all of them, choosing independently random polynomials $\{p_i\}$ and distributing the sequence $\{p_i(x)\}$ along with the evaluation point x.

- If, for some particular i, f_i is constant across our k initial shares $F_{x_1}, \dots, F_{x_k}$, Lagrange interpolation will cause the same constant to appear in the same position for all derived shares. So you can have, say, a fixed header on all of your shares which will be preserved by the secret-sharing mechanism.

- Similarly, for some particular i, you set $f_i = x$, i.e. you encode the evaluation point in a fixed place in your sequence, then Lagrange interpolation will interpolate the polynomial $p(x) = x$ here and place the correct value of x in the correct place for all shares.

- Going even further, suppose that for each initial share $F_x = \{f_i\}$, a particular character can be described as a particular **affine transformation** of the others, like

  $$f_j = \sum_{i \neq j} \alpha_i f_i + \beta$$

  for a fixed index j and fixed $\beta, \alpha_i \in \mathbb{F}$.

  Then *all derived shares will satisfy the same equation*!

This is not immediately obvious but can be shown by interpolating the polynomial

$$q(x) = \sum_{i \neq j} \alpha_i p_i(x) + \beta - p_j(x).$$

By assumption, this polynomial is zero at each evaluation point x and is therefore zero everywhere.

This fact is so important that we term it the **Fundamental Theorem of Computing SSSS with Volvelles**.

Error correcting codes can be characterized in terms of affine transformations. For example, the codex32 error correcting code is computed by adding thirteen extra "checksum" characters to our data, each of which is a particular affine transformation of the other characters.

The Fundamental Theorem therefore implies that if we apply any checksum derived from such a code to our initial shares, that the derived shares will automatically be checksummed as well.

For more information about volvelles, see the next two sections.

2 Volvelles and Tables

The basic tools of hand computation are lookup tables for operations in $\mathbb{F}_{32}$. With only 32 elements, we can represent binary operations using reasonably-sized 1024-element tables.

The four basic operations are provided in the booklet as "Principal Tables" and also implemented as **volvelles**, which are simple computers constructed by two sheets of paper, cut into circles and affixed at the center so that they are able to rotate relative to each other. We frequently refer to volvelles just as *wheels*.

This section explains the underlying operations, the encodings, and the volvelles.

2.1 The Bech32 Alphabet

The previous section indicated that if $\beta \in \mathbb{F}_{32}$, then we can write

$$\beta = b_4\alpha^4 + b_3\alpha^3 + b_2\alpha^2 + b_1\alpha + b_0$$

where each $b_i \in \{0, 1\}$ and the choices for b_i are unique. We can therefore encode β as a 5-bit number by directly encoding the bits b_i. Alternately, since there are only 32 such βs, we assign them all alphanumeric symbols, with four symbols to spare. This is the premise behind the **bech32 alphabet**, defined in BIP 173, and reproduced on the following page.

In addition to the bech32 alphabet, which uses Latin characters, we also use an alternate alphabet using Greek letters and various symbols.

We have ordered all the symbols in three ways — αbetically, alphabetically, and by their "numeric" binary value. These three representations are useful in different contexts:

1. Representing elements as a power of α makes multiplication very easy, since multiplication is just addition mod 31 in the exponent.

   This is how our multiplication wheel can be implemented as a circular slide rule.

2. Representing alphabetically makes it easy for humans to scan and sort.

3. Representing in binary is how the elements are typically stored in computers, can be used to convert data from other encodings. Addition is simply xor in this format.

The following page enumerates all 32 elements of $\mathbb{F}_{32}$, ordered by their different encodings. This page can be used as a reference when doing operations not supported by the volvelles. (All the operations in the booklet, including checksumming, secret splitting and recovery, can be done with the volvelles. But readers of this document may want to play with the algebra more directly, in which case they will need this table.)

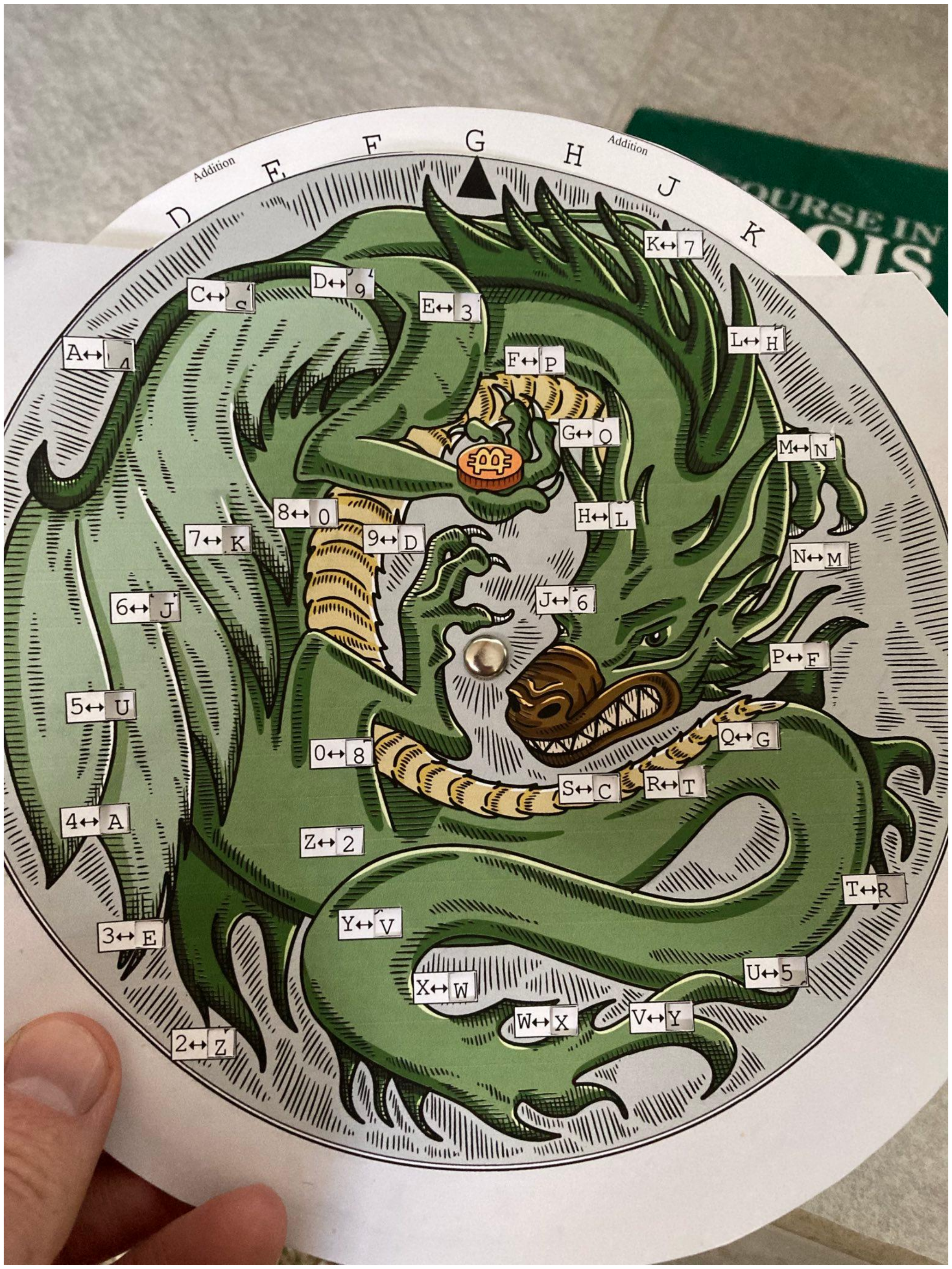
Q	$\times$	-	00000
P	$\aleph$	α^0	00001
Z	α	α^1	00010
Y	Γ	α^2	00100
G	Θ	α^3	01000
S	Ψ	α^4	10000
F	Λ	α^5	01001
J	@	α^6	10010
D	ρ	α^7	01101
6	†	α^8	11010
A	¶	α^9	11101
N	#	α^{10}	10011
0	Φ	α^{11}	01111
7	♦	α^{12}	11110
4	¢	α^{13}	10101
R	β	α^{14}	00011
X	ϵ	α^{15}	00110
V	Π	α^{16}	01100
C	¤	α^{17}	11000
E	$\oplus$	α^{18}	11001
M	‡	α^{19}	11011
L	♥	α^{20}	11111
H	€	α^{21}	10111
8	η	α^{22}	00111
W	Σ	α^{23}	01110
U	§	α^{24}	11100
3	Ω	α^{25}	10001
T	Ξ	α^{26}	01011
K	¥	α^{27}	10110
9	Δ	α^{28}	00101
2	μ	α^{29}	01010
5	%	α^{30}	10100

A	α^9	11101
C	α^{17}	11000
D	α^7	01101
E	α^{18}	11001
F	α^5	01001
G	α^3	01000
H	α^{21}	10111
J	α^6	10010
K	α^{27}	10110
L	α^{20}	11111
M	α^{19}	11011
N	α^{10}	10011
P	α^0	00001
Q	-	00000
R	α^{14}	00011
S	α^4	10000
T	α^{26}	01011
U	α^{24}	11100
V	α^{16}	01100
W	α^{23}	01110
X	α^{15}	00110
Y	α^2	00100
Z	α^1	00010
0	α^{11}	01111
2	α^{29}	01010
3	α^{25}	10001
4	α^{13}	10101
5	α^{30}	10100
6	α^8	11010
7	α^{12}	11110
8	α^{22}	00111
9	α^{28}	00101

Q	-	00000
P	α^0	00001
Z	α^1	00010
R	α^{14}	00011
Y	α^2	00100
9	α^{28}	00101
X	α^{15}	00110
8	α^{22}	00111
G	α^3	01000
F	α^5	01001
2	α^{29}	01010
T	α^{26}	01011
V	α^{16}	01100
D	α^7	01101
W	α^{23}	01110
0	α^{11}	01111
S	α^4	10000
3	α^{25}	10001
J	α^6	10010
N	α^{10}	10011
5	α^{30}	10100
4	α^{13}	10101
K	α^{27}	10110
H	α^{21}	10111
C	α^{17}	11000
E	α^{18}	11001
6	α^8	11010
M	α^{19}	11011
U	α^{24}	11100
A	α^9	11101
7	α^{12}	11110
L	α^{20}	11111

2.2 The Addition Wheel

Our first volvelle, the Addition Wheel, is a **slide chart**, which is essentially a lookup table. The top sheet has 32 holes cut into it, one for each character of $\mathbb{F}_{32}$, and an index pointer. The bottom sheet has 1024 results, of which 32 are revealed at each index.



This volvelle computes addition in $\mathbb{F}_{32}$. To compute $x + y$, rotate so that the pointer is pointing at either x or y, then look up the other one on the front page. It is instructive to observe that the expected symmetries are there: $x + y = y + x$, $x + x = Q$, $x + Q = x$, etc.

Volvelles and Algebraic Structure. The addition wheel has 32 holes cut in the face, corresponding to the 32 bech32 characters. If all 32 rotations of the volvelle revealed distinct locations on the bottom wheel, it would require 1024 symbols to be printed on the bottom wheel — but if there were algebraic structure relating the results of different volvelle positions, we could reduce this number.

We will return to this idea in the next section, about slide rules, but for now we simply observe that we did **not** reduce the number of symbols from the maximum 1024.

Why not? Well, observe that the way to reduce symbols is to have two windows at the same radius from the center of the volvelle. Then on the bottom sheet, a single circle of values would provide the revealed symbols for both windows. Let's say that one window is labeled $y \rightarrow$, and the other labeled $z \rightarrow$. Then since the windows are at a fixed angle θ from each other (being printed on the same solid sheet of paper), we would require the bottom circle of values to satisfy

$$\text{for all } x \in \mathbb{F}_{32} : \qquad x + y \text{ and } x + z \text{ are at angle } \theta \text{ to each other}$$

Note that if x ranges over all values in $\mathbb{F}_{32}$ then so does $x + y$. Furthermore, we have $x + z = (x + y) + (y + z)$ (recall we are in characteristic 2). Thus substituting $x + y$ in place of x gives us

$$\text{for all } x \in \mathbb{F}_{32} : \qquad x \text{ and } x + (y + z) \text{ are at angle } \theta \text{ to each other}$$

Since $x + (y + z) + (y + z) = x$, two applications of the above condition gives us

$$\text{for all } x \in \mathbb{F}_{32} : \qquad x \text{ is at angle } 2\theta \text{ from itself}$$

It is now clear that we either need to repeat characters (defeating the goal of reducing the amount of symbols on the bottom wheel) or have $\theta = 180^\circ$.

Okay, so perhaps we can get a 50% reduction in density for the bottom wheel, by setting $\theta = 180^\circ$ and having the windows on opposite sides of the top wheel be at the same radius and use the same set of bottom-wheel symbols.

Let's play this out. Suppose we place the `A` and `T` windows at the same radius on opposite sides of the top wheel. Note that `A` and `T` differ by `K`. Now suppose that when the pointer is at some arbitrary symbol x, the two opposite windows at `A` and `T` show $x + A = y$ and $x + T = z$. Adding these two equations gives $y + z = A + T = K$.

In other words, for this compression to work with the choice of `A` and `T` windows being at the same radius, we need every pair of opposing symbols to add to `K`; i.e. we need to take the sixteen 2-element cosets obtained by modding out by `K` and then order the symbols so that each coset's members appear opposite each other.

It can be seen, by modding out by every possible symbol, and trying various orderings of the resulting cosets, that no such choice will lead to a "natural" ordering[4]. This means that to get this compression, we'd need to reorder the wheel such that users wouldn't know which direction to spin to find a desired symbol, and the resulting harm to usability would exceed the benefit of having larger windows.

If this argument was too abstract, take the addition volvelle and spin it to `C` (one right of `A`) and look at the symbols in the `A` and `T` windows. Then spin it 180° to `U` (one right of `T`) and look at the same symbols. You will see different symbols. For this scheme to work, they would need to be the same symbols. Ergo, we'd have to reorder the symbols in a confusing order to make this work.

(By the way, this *could* be made to work if we rearranged our mapping between bech32 symbols and $\mathbb{F}_{32}$ objects so that the `K`-cosets, or whatever, were naturally ordered. But deviating from the bech32 spec in this way, for such a minor benefit in volvelle layout, doesn't seem worth the potential confusion/incompatibility between the schemes.)

2.3 The Fusion-Translation Wheel

Fusion. While the addition volvelle could not be re-arranged to reduce the number of symbols beyond 1024, let's consider the second operation we might like to do: *multiplication.*

For reasons that we will describe later, when multiplying in $\mathbb{F}_{32}$ it turns out that we want to use the alternate symbol alphabet rather than the bech32 alphabet. We also don't care so much about multiplication by zero, which always results in zero, which we can tell the user rather than putting it into a volvelle.

Now, we have 31 nonzero elements, so a volvelle would naively have $31^2 = 961$ entries. Can we do better? Using the same reasoning as with the addition volvelle, if we wanted two windows $y \to$ and $z \to$ to share a radius, we'd need that

$$\text{for all } x \in \mathbb{F}_{32}: \qquad xy \text{ and } xz \text{ are at angle } \theta \text{ to each other}$$

We have a group under multiplication with 31 elements in it. Since 31 is prime, it is then a fact that if we choose any element $\alpha \in \mathbb{F}_{32}^*$ except 1, that α **generates** the group. Meaning that every element x, including 1, can be written as $z = \alpha^{i_x}$ where i_x is some integer modulo 31. So we may write

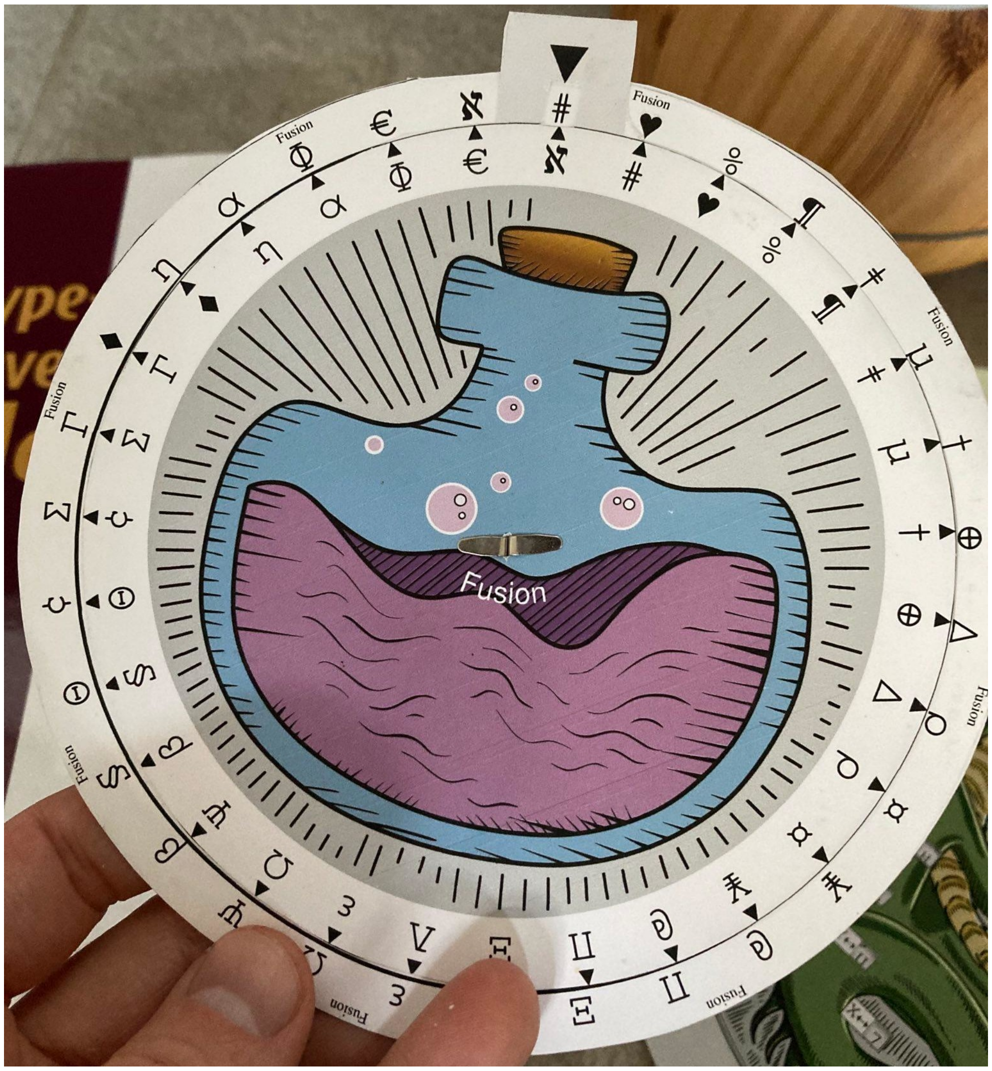
$$\text{for all } \alpha^{i_x} \in \mathbb{F}_{32}: \qquad \alpha^{i_x}\alpha^{i_y} = \alpha^{i_x+i_y} \text{ and } \alpha^{i_x}\alpha^{i_z} = \alpha^{i_x+i_z} \text{ are at angle } \theta \text{ to each other}$$

By squinting at this for a moment, you can observe that if θ is one 31th of a full rotation, and we make sure that each α^i on the front wheel is followed by α^{i+1}, then *every single window can have the same radius.* In fact, we don't need windows, since the bottom wheel

[4]There are 16 cosets, so $15! \approxeq 2^{40}$ different arrangements around a circle. Then you can exchange the members in each coset, for another 2^{15} possibilities. So an exhaustive search would require about 2^{55} work. I did not do an exhaustive search, so I may be wrong in claiming that "no such choice" works. But I spent several hours starting from random permutations and then looking for local optima and never got very close. My measure of "naturalness" was to take the distance d between each character and its alphanumeric successor, and to sum all the 2^ds.

now has only a single circle of symbols, all of which are always visible.

This is the intuition behind the **multiplication wheel**, which is actually a **circular slide rule**:



We use the term **fusion** rather than *multiplication* because we were concerned that by having wheels labeled both "addition" and "multiplication", that users may use their intuition about these operations on the integers, and take incorrect shortcuts.

Translation. There are actually two kinds of multiplication that we might want to do: symbol-by-symbol multiplication and symbol-by-bech32-character multiplication. The former we called fusion, and latter we refer to as **translation**.

Algebraically, fusion and translation are identical, of course. But in practice, fusion is used to multiply $k = 2$ Lagrange basis polynomials (encoded as symbols) to get $k > 2$ basis polynomials (also symbols). Meanwhile transalation is used to multiple the basis polynomials (symbols) by share values (characters) to get translated shares (characters).

So by using different names for the wheels, and different encodings of the underlying field elements, we have implemented a "type system" which makes it hard for users to do operations in an incorrect order.

Since translation is identical to fusion, we might hope that we could construct the translation slide rule by simply relabelling the fusion one. Indeed, we could do this by changing the inner wheel to use bech32 characters rather than symbols. Then to translate a character c by a symbol σ, for example, the user would turn the wheel to point to c, look for σ, and find what it points to.

There are a couple minor issues with this approach:

- Because the correspondence between bech32 characters and $\mathbb{F}_{32}$ does not have any algebraic structure, ordering alphanumeric characters by increasing powers of α results in an unintuitive ordering.

  We have chosen to just live with this problem. All the characters are visible at the same time, so it isn't nearly as a bad a usability burden as it would've been with a volvelle.

- Since zero (`Q`) is a valid share value, we actually do need to think about multiplication by 0. This differs from the fusion case, since zero is not a valid Lagrange basis polynomial. (0 can only be obtained by trying to use the `S` share as an *input* rather than *output*. But as we will see in the next section, the Recovery Wheel won't let you do this.)

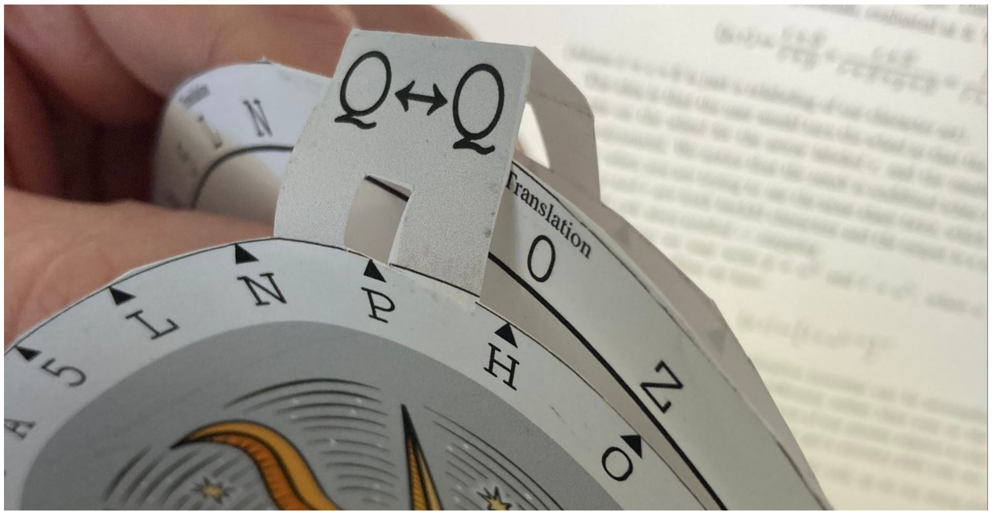
  We solve this by just printing `Q` $\leftrightarrow$ `Q` on the handle of the slide rule.

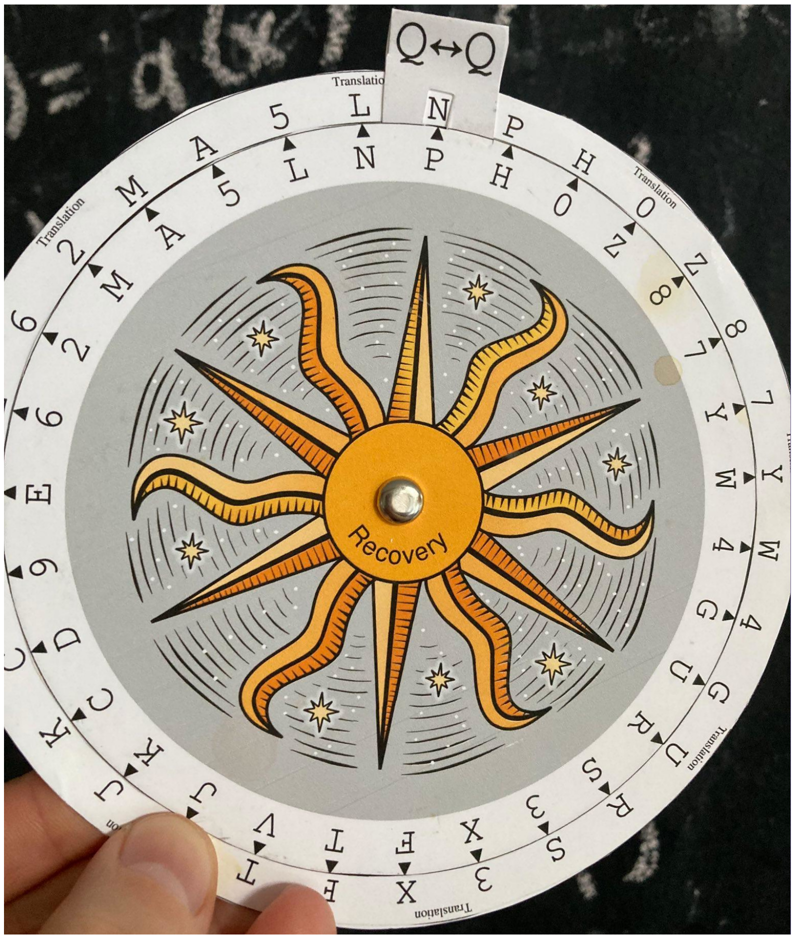
The biggest issue with just relabeling the inner wheel is that the user is trying to map bech32 characters to bech32 characters, but indexing this mapping by a symbol. So if she wants to translate a share by σ say, for all 48 characters in her share, she'll need to rotate the wheel to the input character then check where σ points to find the output character.

This is tiring and error prone. It would be better if she could just turn the wheel to σ and then look up every character without further rotations. How can we achieve this?

After several abortive attempts to add a third circle of characters to the volvelle, Leon had the idea to put the Translation Wheel *on the back of the Fusion Wheel.* Then the user can use the Fusion side to index the mapping, then flip over the wheel and use the Translation side to actually do the mapping!

Of course, we are mathematicians, so while we have brass fasteners, we have no glue. So the actual assembly method is to print both sides attached to each other, then fold them together. We then have two slide rules, whose top and bottom wheels are now the "outer" and "inner" wheels, and which are on opposite sides of the same pages.





(The wheel in the photo is incorrectly labeled "Recovery", because I messed up the PostScript and didn't notice before printing.)

2.4 The Recovery Slide Rule

There is one remaining paper computer to design. This one is the **Recovery Wheel**, which computes Lagrange basis polynomials, evaluated at `S`. That is, it computes the map

$$(p, r) \mapsto \frac{r + S}{r + p} = \frac{r + S}{r + S + p + S} =: \frac{\hat{r}}{\hat{r} + \hat{p}} = \frac{1}{1 + \hat{p}/\hat{r}}$$

(where $\hat{c} \coloneqq c + \texttt{S}$ is just a relabeling of our character set).

Note that both inputs are bech32-encoded share indices, while the output is a symbol-encoded Lagrange basis polynomial evaluated at `S`.

The idea is that the user would turn the wheel so that the pointer points at share index p, looks on the wheel for the arrow labeled r, and the resulting symbol is the Lagrange basis polynomial. We notice that the result is undefined when $r = p$, which corresponds to the case when you are trying to use the same share twice, which makes intuitive sense.

Now, as before we may write $\hat{p} = \alpha^{i_{\hat{p}}}$ and $\hat{r} = \alpha^{i_{\hat{r}}}$, where α is a generator of our multiplicative group. Then we have

$$(p, r) \mapsto \left[1 + \alpha^{i_{\hat{p}} - i_{\hat{r}}}\right]^{-1}$$

The addition of 1 and the multiplicative inversion can be accomplished by more relabeling, and the fact that we have a difference rather than sum in the exponent of α can be accomodated by reversing the direction of our arrows relative to the arrows on the multiplication wheel.

As with the Translation Wheel, we are indexing by a different set than either our input or output. However, the Recovery Wheel only needs to be used once per share, so it is fine for it to be a bit less convienent to use. The real usability concern is that the use of bech32 characters for both indexing and output makes it quite easy to accidentally use the wheel backward.

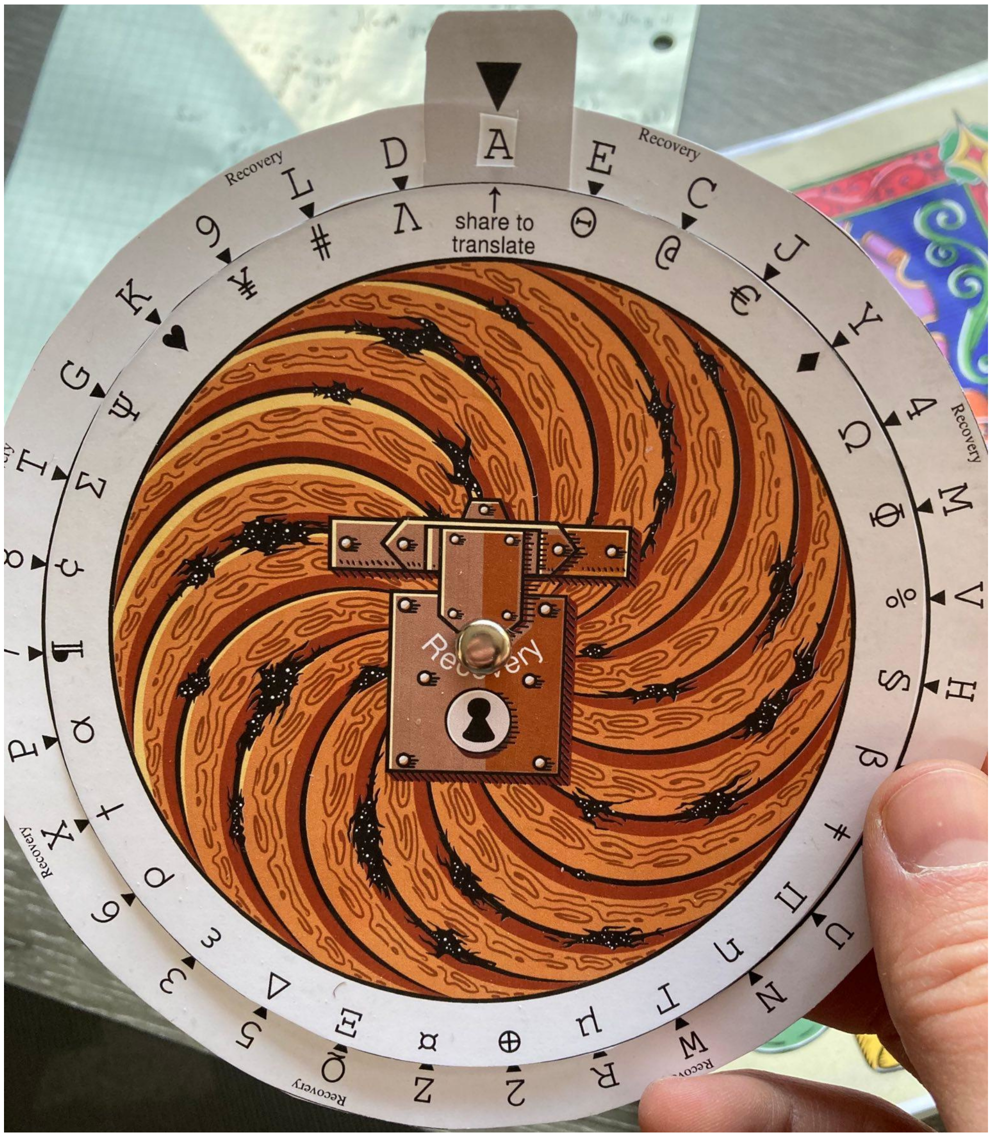
Putting it all together, to get a recovery slide rule, we

1. Reverse the arrows in our multiplication slide rule, as we are doing division rather than multiplication.

2. Apply $x \mapsto x + \texttt{S}$ to the inputs (which are now on the bottom wheel).

3. Apply $x \mapsto [1 + x]^{-1}$ to the output (on the top wheel).

4. Since 0 is not a possible output value, there is a blank space on the top wheel, which conveniently is located directly below the handle. We can take advantage of this blank space to label the handle "share to translate", which hopefully will guide users to use the device correctly.



Finally, we observe that, if we center our slide rule on the α^0 output slot (the one with the pointer on it, rather than a symbol), looking j positions to the left and right we have the values

$$\left[1+\alpha^{-j}\right]^{-1} \text{ and } \left[1+\alpha^{j}\right]^{-1}$$

and it can be computed directly that these sum to 1. In fact, this is exactly the relation between pairs of Lagrange basis polynomials for the case $k = 2$. (There are only two of them, and they add up to one because Lagrange basis polynomials form affine sets.) So on the plain (non-art-covered) version of this wheel, we can draw horizontal lines between these pairs, making the wheel visually distinct from the other plain wheels and also guiding the user in the $k = 2$ case.[5]

[5]This was another innovation from Leon, who observed that to look up the basis polynomials for $k = 2$, you turn to one symbol and look up the other; then swap the symbols and repeat. So by some sort of

The two alphabets. Now that we have all four of our paper computers defined, we can see the justification for having two alphabets: *share data* and *share indices*, which are secret and need to be stored, are represented by bech32 characters, while *Lagrange multipliers*, which are not secret and never stored, are represented by symbols.

Because share indices are actually encoded as share data, these arguably-distinct kinds of data must be encoded the same way. There does not seem to be any ergonimic way to avoid this.

This user-centric categorization is conveniently reflected in the operations that need to be done:

- Share data may be added to other share data, but it is never multiplied by other share data, only by Lagrange basis polynomials (i.e. "translation").
- Lagrange polynomials never added to each other, only multiplied by each other (i.e. "fusion") or by share data (i.e. "translation"). In these two cases the output data is different.
- To do share derivation or recovery, a user cannot even start without obtaining a Lagrange multiplier, which will come either from the Recovery Wheel or from tables provided in the booklet. This ensures the user has obtained all the necessary data before being able to start secret sharing.

More explicitly, both share creation and secret recovery are implemented as Lagrange interpolation using Equation (1) on page 4. In that equation, we are performing a linear combination of y_i values, which are share data (bech32 characters), multiplied by evaluated Lagrange basis polynomials $\ell_i(x)$s (symbols).

The Lagrange basis polynomials are always products of the form $(x + y)/(x + z)$. The Recovery Wheel and Fusion Wheel allow the user to compute these products when they are evaluated at `S`, regardless of which of the combinatorially-many sets of shares they may have started with. During share derivation, the complete products are pre-computed and provided in tables, and require the user to begin with a prescribed set of initial shares.

3 BCH Codes

Now we have our mathematical foundation, from field arithmetic to Lagrange interpolation, and have seen how to express this in volvelles. The final component of our scheme is the checksum, which is a **BCH code**. There is a rich and enormous theory underlying BCH codes, and linear codes in general, but we will take an operationalist point of view and summarize just the facts that we need:

symmetry of turning, the resulting outputs will be pairs of symbols opposite each other.

I'm not sure I see the symmetry he was referring to, but there is a simpler algebraic reason: when you exchange r and p, as you do when computing the two Lagrange basis polynomials for the $k = 2$ case, you replace $\hat{r}/\hat{p}$ with its reciprocal, which is the same as mirroring it over the α^0 position on a slide rule. Then the fact that these add to 1 is just a restatement of the fact that Lagrange basis polynomials form an affine set.

An **error-correcting code** is a mapping from a set of raw data, called **messages** into a larger set of **codewords** which have some extra algebraic structure. In particular, between every pair of codewords there is a minimum **distance** d, which measures the number of characters at which the two codewords differ.

Since all codewords differ by at least the minimum distance d, if there are at most $d - 1$ errors in a codeword, the result is guaranteed not to be a valid codeword, and to be detected as an error pattern. If there are fewer than $d/2$ errors, the correct codeword is uniquely determined by being the closest codeword to the error pattern, which means that this many errors can, in principle, be corrected.

A **BCH code** is a specific type of error-correcting code. In a BCH code, codewords are constructed by encoding data as the coefficients of a large polynomial, then affixing some number of **checksum characters**. The checksum characters are chosen so that the full result is in a specific residue class modulo a **generator polynomial** $G(x)$.

Given a potential codeword, there is a unique minimum-degree polynomial equivalent to this codeword modulo $G(x)$, which can be found by modular reduction. We refer to this reduced polynomial as the **residue** of the codeword.

There are several properties of a BCH code that we will need:

- The **degree** of the code is the maximum degree of the roots of its generator polynomial. (Since the generator polynomial is not, in general, irreducible, this is *not* the degree of the generator polynomial.)

  The degree of our BCH codes, and that of bech32, is 2. Degree-1 BCH codes are called **Reed-Solomon codes**.

  The degree indicates the dimension of the extension field you need to work in to do error correction. So for our codes, you would need to work in a quadratic extension of $\mathbb{F}_{32}$, i.e. $\mathbb{F}_{1024}$.

- The **length** of the code is how long a coded message can be (including the checksum!) for the code to retain its error-correcting properties.

  The length ℓ can be computed as the smallest polynomial of the form $x^{\ell} - x$ that the generator divides. This can be seen by observing that if you have a message of this length, that the ℓth character will be interpreted as the coefficient of $x^{\ell-1} \equiv 1 \mod G(x)$, meaning that it will just mask your first character rather than being checksummed independently.

  We have two codes – the "normal" codex32 has a length of 93. For 512-bit seeds, we need more than 93 characters, so we use an alternate code of length 1023. The length of bech32 is also 1023 — although through an exhaustive search, it was found to have better error detection properties than the algebra would suggest, up to length 71.

- The m-value, or **target residue**, is the specific residue that all codewords must must have modulo G. All m values are equivalent, in the sense that it is easy to convert from one m value to another (simply add the difference to the checksum characters of every codeword).

But $m = 0$ has the particularly bad property that any codeword can be extended by addition of an arbitrary number of 0s to get another valid codeword. (This highlights the fact that BCH codes are designed to handle only *substitution* or *erasure* errors, not insertions or deletions.) Other small values of m have similar issues; bech32 was originally defined to have $m = 1$ but later needed to be modified to bech32m for this reason. bech32m uses a large m instead[6].

On the other hand, $m = 0$ makes a BCH code a **linear code**, and brings with it a ton of algebraic properties which are needed for analysis, so this is what is used in the literature.

In practice it is common to use a string of all-bits-one for m. For our code, we chose characters which spell out `SECRETSHARE32` for the standard code, and `SECRETSHARE32EX` for the alternate one.

- The **checksum length** is the number of extra characters that need to be added to a string to ensure that it has the correct residue. This value *is* the degree of the generator polynomial.

  Our standard code, of length 93, has a checksum length of 13. The alternate code has a checksum length of 15.

We will return to BCH codes in the next section, when we discuss the Checksum Worksheet and the properties of BCH codes which make them compatible with Shamir's Secret Sharing.

3.1 The `codex32` Checksum

We now get into the meat of the document, where we descibe how the actual user processes are implemented. We start with the checksum.

Our primary checksum is defined by a BCH code with generator polynomial

$$\begin{aligned} G(x) = x^{13} &+ \mathtt{E}x^{12} + \mathtt{M}x^{11} + \mathtt{3}x^{10} + \mathtt{G}x^{9} + \mathtt{Q}x^{8} + \mathtt{E}x^{7} \\ &+ \mathtt{E}x^{6} + \mathtt{E}x^{5} + \mathtt{L}x^{4} + \mathtt{M}x^{3} + \mathtt{C}x^{2} + \mathtt{S}x + \mathtt{S} \end{aligned}$$

Our alternate checksum has generator polynomial

$$\begin{aligned} \hat{G}(x) = x^{15} &+ \mathtt{0}x^{14} + \mathtt{2}x^{13} + \mathtt{E}x^{12} + \mathtt{6}x^{11} + \mathtt{F}x^{10} + \mathtt{E}x^{9} \\ &+ \mathtt{4}x^{8} + \mathtt{X}x^{7} + \mathtt{H}x^{6} + \mathtt{4}x^{5} + \mathtt{X}x^{4} + \mathtt{9}x^{3} \\ &+ \mathtt{K}x^{2} + \mathtt{Y}x^{1} + \mathtt{H} \end{aligned}$$

[6]For more details, see `https://gist.github.com/sipa/14c248c288c3880a3b191f978a34508e`

Because the alternate code has substantially the same properties as the primary one, and the primary one is easier to work with, we will only focus on the primary one from here on.

To interpret the $\mathbb{F}_{32}$ elements such as `E`, `M`, etc., recall that there is a table on page 8 containing all the elements and their alternate encodings.

To checksum a string of bech32 characters $\{v_i\}_{i=1}^n$, we want to encode it as a polynomial

$$p(x) = x^n + \sum_{i=0}^{n-1} v^{n-i} x^i$$

such that $p(x) \bmod G(x)$ is equal to `SECRETSHARE32`, i.e.

$$\mathtt{S}x^{12} + \mathtt{E}x^{11} + \mathtt{C}x^{10} + \mathtt{R}x^9 + \mathtt{E}x^8 + \mathtt{T}x^7 + \mathtt{S}x^6 + \mathtt{H}x^5 + \mathtt{A}x^4 + \mathtt{R}x^3 + \mathtt{E}x^2 + \mathtt{3}x + \mathtt{2}.$$

Remembering throughout that an n-degree polynomial has $(n + 1)$ terms, the way that we achieve this for an arbitrary message $\{v_i\}$ is to concatenate the target residue `SECRETSHARE32` to $\{v_i\}$ as

$$\hat{p}(x) = x^{n+13} + \sum_{i=0}^{n+13-1} v^{n+13-i} x^i \bmod G(x) \tag{2}$$

We call the thirteen resulting characters the **checksum** and replace our copy of `SECRETSHARE32` with them.

Why does this work? Basically, to get our residue to match some specific target, we just need to subtract the existing residue from our string, then add the target. Since both the existing residue and the target value have degree 12, this will affect only the rightmost 13 charactes of our string.

So we multiply by x^{13}, moving our actual data to the left 13 spaces, and add `SECRETSHARE32`. Then the above instructions "subtract the existing residue from our string, then add the target" are easy. First we add the target, which since we are in characteristic 2, means zeroing out the `SECRETSHARE32` we just added by adding another one. Then we subtract the existing residue from this 0, which (again because of characteristic 2, where negation is a no-op) means just copying it into place.

3.2 The Checksum Worksheet

As described in the last section, our checksum verification algorithm is: encode the message as a polynomial, take it mod $G(x)$, and compare it against the fixed string `SECRETSHARE32`. The checksum generation algorithm is essentially the same: append `SECRETSHARE32` in place of the checksum, take the resulting polynomial mod $G(x)$, then use the result as the actual checksum.

How do we do this in practice? In short, we build up our polynomial two characters as a time: multiply by x^2, add two new characters, reduce, repeat. In detail:

Checksum Verification.

1. Take the prefix `ms1` and encode it as a bech32 human readable part (HRP), which works out to `RRQDN` (the process is to take the high 3 bits of each 8-bit ASCII character, followed by 0, followed by the low 5 bits of each character). Prefix a 1, or `P` in bech32. The resulting initial polynomial is `PRRQDN`, or

   $$x^5 + \mathtt{R}x^4 + \mathtt{R}x^3 + \mathtt{D}x + \mathtt{N}$$

   Multiply this by x^{13} and reduce it mod $G(x)$. The result will be `33XW87RRYLJG`. This string is initially filled in in the checksum worksheet.

2. Fill in the first 13 characters of the data to be checksummed. Add this to the pre-filled values. The result will be the residue of `ms1<first 13 user characters>`, and this is what the first three lines of the checksum worksheet compute.

3. Multiply by x^2 and fill in the next two characters of data. The multiplication is done by shifting the characters two spaces to the left, equivalently, shifting the entire rest of the worksheet 2 spaces to the right. This accounts for the diagonal shape of the worksheet.

   The resulting 3rd row has 15 characters, where the leftmost characters ℓ_1 and ℓ_2 are the coefficients of x^{13} and x^{14}.

4. Reduce the whole string mod $G(x)$: first, compute the reduction of the leftmost characters, $\ell_1 x^{14} + \ell_2 x^{13}$. This is a nontrivial computation, so we simply provide a giant "Checksum Table" in which the user can look up ℓ_1 and ℓ_2.

   Copy the table entry into line 4. Then add the remaining 13 characters, which are unaffected by reduction since G has degree 13, to this.

   It may be instructive to read the PostScript code for the Checksum Table, which enumerates all the two-character possibilities, and for each one, computes the reduction mod G.

5. Repeat the previous two steps: add the next two characters of data to the right of the just-filled-in line, lookup the leftmost two characters in the Checksum Table to fill in the next line, and add the results.

   Repeat until you run out of data.

6. Check that the final result is `SECRETSHARE32`.

Checksum Generation. This is basically identical to checksum verification, except that the final 13 characters are initially not available. On the worksheet these are colored pink to indicate to the user that she should stop filling random data into the cells.

In the mathematical description we suggest concatenating `SECRETSHARE` to the data itself, then replacing it. It is equivalent to instead write `SECRETSHARE` in the in the bottom-most row (whose cells lie below the final 13 data cells), and then "backsolve" by subtracting (i.e. adding) all the rows above it. This will place the residue, our desired checksum, in the correct place.

4 Secret Sharing

In the Mathematical Preliminaries section we described Lagrange interpolation and Shamir's Secret Sharing Scheme (SSSS). We defined the Fundamental Theorem of Computing SSSS With Volvelles as the observation that Lagrange interpolation allows evaluating a polynomial at a fixed point as an affine combination of its evaluations at other fixed points.

As a consequence, any linear or affine relationship between the original evalutations will be preserved. We will come to this in a moment, but first let's describe the SSSS process.

4.1 Computing SSSS

There are two places where we use Lagrange interpolation:

- When creating a share with index x, we start with k initial shares x_1, x_2, …, x_k whose indices are always the first k symbols of the bech32 alphabet A, C, D, etc.

  (There is an alternate process where the user starts with a fixed secret, in which case x_1 is S and the other x_i's are shifted, but the rest is exactly the same.)

- When recovering a secret, we start with k shares with indices $\{x_i\}_{i=1}^k$ which are fixed during recovery but unpredictable in advance, and compute the S share.

In both cases, the computation is straightforward: evaluate (1) as

$$p(x) = \sum_{i=1}^{k} \ell_i(x) y_i$$

Here y_i is the data of the ith share and $\ell_i(x)$ is a Lagrange multiplier which must be computed by the user. We refer to these multiplies as "Recovery Symbols" in the text. To avoid confusion, they are always encoded using the symbol alphabet rather than bech32. The process for obtaining these is:

- During share creation, the evaluation points $\{x_i\}$, which are the user's initial share indices, are always fixed. Therefore $\ell_i(x)$ is purely a function of x (the index of the share to be created) and k. There aren't that many possibilities so we simply provide tables on the "Constructing Shares" page.

  (It is instructive to modify the PostScript source for these tables to allow "deriving" the initial shares from themselves. You will find that the index for the initial share under question becomes 1 (ℵ) while the index for all the other shares is 0 (×).)

- During recovery, there are up to 31 outstanding non-S shares and the user has an arbitrary subset of k of them. There are $\binom{31}{k}$ possibilities, which for $k = 2$ is a reasonable number — 465 — but for $k \geq 3$ quickly grows out of hand, to 4495, then 31465, and beyond.

For $k = 2$ we provide a table, both in tabular form and in the form of a volvelle. This is what the Recovery Wheel ultimately is, and at position p, window w it computes

$$\ell_i(\mathtt{S}) = \frac{w + \mathtt{S}}{w + p}$$

Here position p is the index of share y_i whose multiplier is being computed, and w is the index of the other share. To avoid getting p and w confused, we have written "Share to Translate" on the handle of the disc.

For $k \geq 3$, for target share p (index of y_i) with other shares $\{w_i\}_{i=1}^{k-1}$, we need to compute

$$\ell_i(\mathtt{S}) = \prod_{i=1}^{k-1} \frac{w_i + \mathtt{S}}{w_i + p}$$

with notation chosen to highlight that *the Lagrange basis polynomials for $k \geq 3$ are products of the basis polynomials for $k = 2$.*

This fortuitous fact means that the user can compute polynomials $\ell_i(\mathtt{S})$ by looking up factors using the Recovery Wheel (rotate it to index p then read all the w_i's off the front), and then multiplying all the results using the Fusion Wheel. (Turn the wheel so it points to the first symbol, then look up the second. Turn it to whatever the second symbol is pointing to, look up the third, and so on.)

The Fusion Wheel is the only one which takes two symbols as input, and multiplication is commutative so the order of inputs doesn't matter, so it is hard for the user to do the wrong thing here.

Once the user has obtained the $\ell_i(x)$'s, the rest is simple: multiply each cell of y_i (the value of the ith share, encoded in the bech32 alphabet) by $\ell_i(x)$ (computed above, encoded in the symbol alphabet) using the Translation Wheel.

Again, the choice of alphabets and commutativity of multiplication make it hard for the user to do the wrong thing.

This will give the user k translated shares $\hat{y}_i$, which she should then add together using the Addition volvelle. We have provided a Translation Worksheet to help keep everything straight.

4.2 SSSS and Checksumming

Once the user has computed her derived share(s) or recovered her secret S share, she will likely notice that the produced share header is of the correct form and has the correct index. This is oddly thrilling but not mathematically surprising: the fixed parts of the header are produced by interpolating a constant polynomial and the share index is produced by interpolating the polynomial $f(x) = x$.

What is more mathematically impressive is that the last 13 symbols of the polynomial will constitute a valid checksum for the resulting share. This is because each checksum symbol is defined as an affine transformation of the other characters, and the Fundamental Theorem says that any such relationships will be preserved.

In fact, in a complete Checksum Worksheet, *every single cell* that the user fills in is an affine function of the share data. This means that if you pick an arbitrary cell, say, the fourth cell of the tenth row, you can use the above process to combine those cells from the initial shares' worksheets and produce the corresponding cell on the derived share's worksheet.

This means in particular, that the final row (`SECRETSHARE32`) will be preserved, meaning that the checksum of the derived share will be correct. But it also means that, if the user uses the checksum worksheet to verify her share translation (which she should!) she can "sanity check" her work by pre-deriving cells.

This is was an exciting realization, because normally the Checksum Worksheet takes a long time to fill out and provides the user no feedback until the very end, at which point the result might just be "wrong residue, start over". To avoid this frustration, we recommend the following process for derived shares and recovery:

1. Before deriving any shares, complete Checksum Worksheets for the inital shares. (During construction these should be readily available; during recovery it's worth doing as a sanity check.)

2. Derive the actual share, using the above method.

3. Copy the share into the top diagonal, i.e. the bolded data cells, of a fresh Checksum Worksheet.

4. Derive the cells of the *bottom* diagonal directly, in the same way that you derived the top diagonal.

5. Start working through the worksheet.

If, at any point, the computed value of a bottom diagonal square doesn't match the precomputed value, it means you made a mistake in that column. Re-derive the top and bottom values and redo the additions before continuing.

5 Quickchecks

Important Note. This section describes a series of worksheets which, as of March 2023, do not actually exist.

Recall that the generating polynomial for our primary code is

$$\begin{aligned} G(x) = x^{13} + \mathtt{E}x^{12} + \mathtt{M}x^{11} + \mathtt{3}x^{10} + \mathtt{G}x^{9} + \mathtt{Q}x^{8} + \mathtt{E}x^{7} \\ + \mathtt{E}x^{6} + \mathtt{E}x^{5} + \mathtt{L}x^{4} + \mathtt{M}x^{3} + \mathtt{C}x^{2} + \mathtt{S}x + \mathtt{S}. \end{aligned}$$

Previously we simply took this polynomial as a given. But in this section and the next, we need to dig a bit deeper into how it was constructed.

To this end we first construct the extension field $\mathbb{F}_{1024}$. Just as we produced $\mathbb{F}_{32}$ by adding a root of the irreducible 5th-degree polynomial $x^5 + x^3 + 1$ over $\mathbb{F}_2$, we can adjoin a root of the irreducible 2nd-degree polynomial $x^2 + x + 1$ to $\mathbb{F}_{32}$ to get a new field $\mathbb{F}_{1024}$. We will call this new root ζ. By construction, ζ satisfies the equation $\zeta^2 = \zeta + 1$.

We can write any element of $\mathbb{F}_{1024}$ as $a + b\zeta$, where a and b are in $\mathbb{F}_{32}$. Multiplication and addition happen in the obvious way, with every ζ^2 factor simply replaced by $\zeta + 1$.

The new field $\mathbb{F}_{1024}$ has 1024 elements; its multiplicative group has $1023 = 3 \cdot 11 \cdot 31$ elements, so unlike the case of $\mathbb{F}_{32}$ where every non-unit element has order 31, in $\mathbb{F}_{1024}$ non-unit elements might have any order in the set $\{3, 11, 31, 33, 93, 341, 1023\}$.

You may recognize the numbers 93 and 1023 as the length of our primary and alternate code (and 1023 as the length of bech32). This is not a coincidence.

Consider the element $\beta = \texttt{G}\zeta$, which has order 93. In fact, our generator G has roots which are all powers of β![7] We can write it as

$$G(x) = \prod_{i \in \{17,20,46,49,52,77,78,79,80,81,82,83,84\}} (x - \beta^i).$$

We can now see why our code has length 93 — all its roots satisfy $x^{93} - 1$, so G is a factor of $x^{93} - 1$, which means that $x^{93} \equiv 1 \bmod G$. So we cannot distinguish the codewords x^{93} and 1, even though they are distance 2 from each other.

The run of 8 consecutive roots is the reason that this code has distance 9, though the reason is not one we can casually state.[8]

Now, recall G has degree 2, so all of its roots have degree at most 2, or equivalently, it can be factored in $\mathbb{F}_{32}$ into factors which are all linear or quadratic. Specifically:

$$\begin{aligned} G(x) = {} & (x + \texttt{T})(x + \texttt{S})(x + \texttt{C}) \\ & \times (x^2 + \texttt{Z}x + \texttt{Y})(x^2 + \texttt{R}x + \texttt{9})(x^2 + \texttt{2}x + \texttt{K})(x^2 + \texttt{W}x + \texttt{X})(x^2 + \texttt{L}x + \texttt{A}) \end{aligned}$$

Recall further that our codewords are defined by their membership in a specific equivalence class modulo G, the equivalence class of `SECRETSHARE32`. By the Chinese Remainder Theorem, this equivalence class is also characterized by sets of equivalence classes modulo

[7]This is no accident — to construct G, we started with 8 consecutive powers of β, took the minimal polynomials of these, and took the least common multiple of these. The exact choice of β and its powers came down to an exhaustive search of which values led us to a code with our desired properties: distance 9, checksum length 13, maximal length, and three repeated coefficients in the generator polynomial, which cause the entries in the checksum table to have repeated digits, which we believe make transcribing easier for human eyes.

[8]It can be easily found online, e.g. on the Wikipedia page for BCH codes.

these factors. In particular, for a given $v(x)$,

$$v(x) \equiv \texttt{2W} \qquad \bmod (x + \texttt{S})(x + \texttt{T}) = (x - \beta^{84})(x - \beta^{81}) \tag{3}$$

$$v(x) \equiv \texttt{LD} \qquad \bmod (x^2 + \texttt{Z}x + \texttt{Y}) = (x - \beta^{83})(x - \beta^{52}) \tag{4}$$

$$v(x) \equiv \texttt{XK} \qquad \bmod (x^2 + \texttt{2}x + \texttt{K}) = (x - \beta^{82})(x - \beta^{20}) \tag{5}$$

$$v(x) \equiv \texttt{9X} \qquad \bmod (x^2 + \texttt{W}x + \texttt{X}) = (x - \beta^{80})(x - \beta^{49}) \tag{6}$$

$$v(x) \equiv \texttt{LT} \qquad \bmod (x^2 + \texttt{L}x + \texttt{A}) = (x - \beta^{79})(x - \beta^{17}) \tag{7}$$

$$v(x) \equiv \texttt{WU} \qquad \bmod (x + \texttt{C})(x + \texttt{T}) = (x - \beta^{78})(x - \beta^{81}) \tag{8}$$

$$v(x) \equiv \texttt{UM} \qquad \bmod (x^2 + \texttt{R}x + \texttt{9}) = (x - \beta^{77})(x - \beta^{46}) \tag{9}$$

if and only if $v(x) \equiv$ SECRETSHARE32 modulo G.

In fact, we have ordered these checks so that if the user checks each one of them, in order, the CRT will fix the equivalence class of her codeword modulo a polynomial $\hat{G}$ with progressively many consecutive roots of β, so she will be guaranteed to detect progressively many errors.

This is the intuition behind the **quickcheck** method of verifying the checksum. Rather than, say, doing a full checksum worksheet every year to verify their codeword module G, the user can instead verify each of Equations (3) through (9), doing one check ever month. Once the final check (9) is done, the user starts back over with (3).

You may notice that the factor $(x + \texttt{S}) = (x - \beta^{81})$ appears twice in this list. For a user doing all the checks, its second appearance is strictly redundant. But by including it, we ensure that no matter where in the list the user is starting from, they consistently accumulate consecutive roots, so that if their data isn't corrupted between individual checks, they gain progressive error detection ability.

Another reason to include $(x + \texttt{S})$ twice is to make all the quickchecks look the same, so that the process is as consistent as possible. The point of this scheme is to encourage the user to frequently engage with their secret data, so that they gain and maintain familiarity with the checksum verification process.

Even more important that consistency, by making each quickcheck use a quadratic generator polynomial, we get two checksum digits, providing 10 bits of protection against random errors. This means that if even a single quickcheck passes, the user has 99.9% assurance (1023/1024) that their data is intact. If we'd used a linear generator polynomial, we would get only 5 bits, so a passing check would give the user only 97% (31/32) assurance.

Each quickcheck is merely a modular reduction of the user's data; it differs from the Checksum Worksheet only in that we are reducing modulo a quadratic rather than modulo the full degree-13 generator. This allows us to rearrange the worksheet in a more aesthetically-pleasing way, and fit the 2-page Checksum Table into a single page (since it is mapping pairs of characters to pairs of characters, rather than pairs of characters to 13-character strings). But the underlying mechanism is exactly the same.

(All of the considerations in this section apply also to our alternate length-1023 code, which is used for 400+-bit seeds. It uses the order-1023 element $\gamma = \texttt{E} + \texttt{X}\zeta$ to generate its roots, in place of β. But since we do not expect anybody to manipulate such large seeds by hand, we will not bother do the equivalent calculations. The motivated user will be able to construct everything merely from knowledge of the generator polynomial and of γ.)

6 Error Correction

Important Note. This section describes a correction table which, as of March 2023, does not actually exist.

Suppose that a user's share data has up to 8 errors in it. We can write their data in polynomial form as

$$\hat{d}(x) = d(x) + e(x)$$

where $d(x)$ is their actual share data, $e(x)$ is some error polynomial with up to 8 terms, and $\hat{d}$ is their actual data.

Taking this modulo G, we see that

$$\hat{d}(x) \equiv d(x) + e(x) \equiv \texttt{SECRETSHARE32} + e(x) \mod G.$$

That is, in the process of modding out by G, the actual secret data is completely erased and is replaced by the target residue. This is why the booklet advises users that they may enter the residue into an electronic computer — it has no secret data in it — and why this is a useful thing to do — it is completely determined by the error pattern.

The task of **error correction** is to undo the reduction process. That is, given a residue $e(x)$ mod G, figure out what $e(x)$ really is. Then the user can recover their share data by computing

$$\hat{d}(x) + e(x) = d(x) + e(x) + e(e) = d(x)$$

Because our checksum has distance 9, if there are up to 8 errors, the reduction is guaranteed not to produce the value `SECRETSHARE32`, so that the error pattern will be recognized as an error.

Furthermore, in all cases where there are 4 or fewer errors, the reduction will produce unique values, so that it can (in principle) be undone.

But doing this is fairly involved. The standard way of doing this is to use the Berlekamp-Massey algorithm to determine an "error locator polynomial", to find roots of this polynomial to determine the error locations, and then to use Forney's algorithm to compute the error values. All of these steps involve arithmetic in $\mathbb{F}_{1024}$ which does not lend itself nicely to volvelles and worksheets. We believe that it possible, but as of March 2023, we have not figured out how.

However, one thing we *can* do is produce a lookup table. In the case that the user has a 48-element share, of which the first 3 elements are definitely the characters `ms1`, then there are 1395 ways in which there may be one mistake in the remaining 45 characters. Each of these 1395 ways produce a unique residue, and we have provided a 3-page table listing them all. Therefore, if the user makes only a single error, they can simply look up their residue to learn how to correct it.

A final observation is that errors might not mean that the share data itself are corrupted. An error in location y actually means that *column y in the Checksum Worksheet was not computed correctly.* This may mean that the top cell, which contains share data, was wrong. But it could also be that the user made an arithmetic error. We therefore advise users, when doing error correction, to completely recompute any erroneous columns of the Checksum Worksheet before they consider modifying their data.

7 Conclusion and Acknowledgements

We thank the Russell O'Connor for noticing the remarkable compatibility between SSSS and BCH codes (or any linear code), which enable user-computed SSSS[9]. Even if SSSS were otherwise tractable to do by hand, without hand-verifiable checksums it would be hopeless for users to notice or recover from arithmetic and transcription mistakes, and this whole project would be unworkable.

We thank the authors of SLIP39, which also uses both SSSS and BCH codes, and inspired us to attempt a hand-computable version of it exploiting the compatibility between the two.

We thank Dr. Curr for then noticing that by using a code over $\mathbb{F}_{32}$ rather than $\mathbb{F}_{1024}$, it is possible to do these computations by hand, and for putting together the initial prototype of this project which included the PostScript fundamentals to do computations with BCH codes, to draw 32-by-32 volvelles, and to do 2-of-n secret sharing.

We thank Micaela Paez for the amazing artwork that adorns the illustrated version of the volvelles.

We thank Peter Todd for his mailing list post in which he suggested replacing the checksum with a single-character one obtained by summing all the share data[10]. Without this we would not have discovered the "quickcheck" method of verifying the checksum.

From that point onward it was a real trip to bring everything together, optimizing the layout of the volvelles and worksheets for user experience, reducing the total number of volvelles, introducing the slide rules, discovering how to localize Checksum Worksheet errors, and adding artwork and color.

The result has been a remarkable, and even mathematically novel, project which fits together much better than any of us expected.

We hope you will appreciate the mathematical beauty of this construction, or at least appreciate the peace of mind that comes with being able to redundantly back up your Bitcoin secrets without the use of electronic computing devices.

Never trust anything that can think for itself, if you can't see where it keeps its brain!
—Arthur Weasley, Harry Potter and the Chamber of Secrets

[9] See `https://lists.linuxfoundation.org/pipermail/bitcoin-dev/2020-August/018070.html`
[10] `https://lists.linuxfoundation.org/pipermail/bitcoin-dev/2023-February/021498.html`